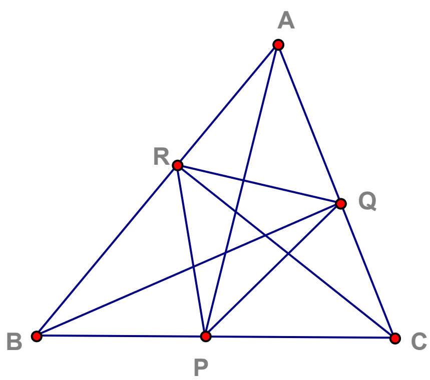
Inequality in a triangle with symedians.
https://www.linkedin.com/feed/update/urn:li:activity:6694859623859580928
JP.300. In $\triangle ABC, I$–incenter, ID, IE, IF –symedians in $\triangle BIC, \triangle CIA, \triangle AIB$, $D \in BC, E \in CA, F \in AB$. Prove that :

$$\frac{[DFE]}{[ABC]} \geq \frac{r^2}{2R^2 - Rr - 2r^2}.$$

Proposed by Marian Urzarescu.
Solution by Arkady Alt, San Jose, California, USA.
First we will consider more general configuration, that is for any three numbbers $\alpha, \beta, \gamma \in (0,1)$ and points $P \in BC, Q \in CA, R \in AB$ such that $\frac{BP}{PC} = \frac{1-\alpha}{\alpha}$, $\frac{CQ}{QA} = \frac{1-\beta}{\beta}, \frac{AR}{RB} = \frac{1-\gamma}{\gamma}$ and we will express $\frac{[PQR]}{[ABC]}$ via α, β, γ.

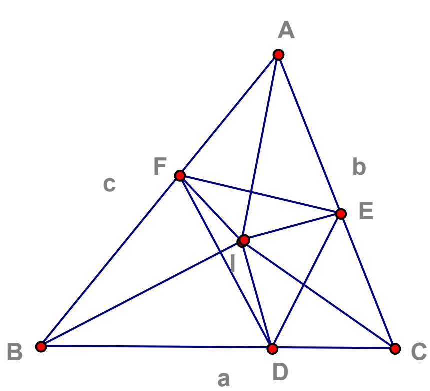


.

We have $[PQR] = \frac{c(1-\gamma) \cdot b\beta}{2} \cdot \sin A = \beta\left(1-\gamma\right) \cdot \frac{bc\sin A}{2} = \beta\left(1-\gamma\right)[ABC]$
and similarly $[BDF] = \gamma(1-\alpha)[ABC]$, $[CED] = \alpha(1-\beta)[ABC]$.
Hence $[PQR] = [ABC](1 - c(1-\gamma) - \gamma(1-\alpha) - \alpha(1-\beta)) =$
$[ABC](1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha) \Leftrightarrow$

(1) $\qquad \frac{[PQR]}{[ABC]} = 1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha.$

Now we can come back to the problem



Let l_a be length of bisector of angle A. Since $l_a^2 = \frac{4bcs(s-a)}{(b+c)^2}$ and $IA = \frac{l_a(b+c)}{2s}$ then $IA^2 = \frac{bc(s-a)}{s} = \frac{abc}{s} \cdot \frac{s-a}{a}$ and similarly we obtain $IB^2 = \frac{abc}{s} \cdot \frac{s-b}{b}$, $IC^2 = \frac{abc}{s} \cdot \frac{s-c}{c}$. Since ID, IE, IF be symedians in $\triangle IBC, \triangle ICA, \triangle IAB$ respectively then $\frac{BD}{DC} = \frac{IB^2}{IC^2} = \frac{\frac{s-a}{a}}{\frac{s-b}{b}}$ and similarly $\frac{CE}{EA} = \frac{IC^2}{IA^2} = \frac{\frac{s-b}{b}}{\frac{s-c}{c}}$, $\frac{AF}{FB} = \frac{IA^2}{IB^2} = \frac{\frac{s-c}{c}}{\frac{s-a}{a}}$

Denoting $x := \frac{s-a}{a}, y := \frac{s-b}{b}, z := \frac{s-c}{c}$ and $\alpha := \frac{y}{x+y}, \beta := \frac{z}{y+z}, \gamma := \frac{x}{z+x}$ we obtain $\frac{BD}{DC} = \frac{1-\alpha}{\alpha}, \frac{CE}{EA} = \frac{1-\beta}{\beta}, \frac{AF}{FB} = \frac{1-\gamma}{\gamma}$ and for this α, β, γ by replacing in (**1**) (P, Q, R) with (D, E, F) we obtain $\frac{[DEF]}{[ABC]} = 1 - \alpha - \beta - \gamma + \alpha\beta + \beta\gamma + \gamma\alpha =$

$$\frac{yzx}{(x+y)(y+z)(z+x)} + \frac{xyz}{(x+y)(y+z)(z+x)} = \frac{2xyz}{(x+y)(y+z)(z+x)}.$$

Let R, r, s be circumradius, inradius and semiperimeter in $\triangle ABC$ tof the problem.

We have $xyz = \frac{(s-a)(s-b)(s-c)}{abc} = \frac{r^2 s}{4Rrs} = \frac{r}{4R}, x+y+z =$

$$\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c} = \frac{s(ab+bc+ca)}{abc} - 3 = \frac{s(s^2+4Rr+r^2)}{4Rrs} - 3 =$$

$$\frac{s^2+4Rr+r^2}{4Rr} - 3 = \frac{s^2-8Rr+r^2}{4Rr},\ xy+yz+zx = \sum \frac{(s-a)(s-b)}{ab} =$$

$$\frac{1}{abc}\sum c(s-a)(s-b) = \frac{1}{abc}\sum(s^2c - (a+b)cs + abc) =$$

$$\frac{2s^3 - 2(ab+bc+ca)s + 3abc}{abc} = \frac{2s^3 - 2(s^2+4Rr+r^2)s + 12Rrs}{4Rrs} = \frac{2R-r}{2R}$$

and $(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz =$

$$\frac{s^2+r^2-8Rr}{4Rr} \cdot \frac{2R-r}{2R} - \frac{r}{4R} = \frac{(2R-r)(s^2+r^2-8Rr) - 2Rr^2}{8R^2 r}.$$

Hence, $\frac{2xyz}{(x+y)(y+z)(z+x)} = \frac{\frac{r}{2R}\cdot 8R^2 r}{(2R-r)(s^2+r^2-8Rr) - 2Rr^2} =$

$$\frac{4Rr^2}{(2R-r)(s^2+r^2-8Rr) - 2Rr^2} \geq \frac{4Rr^2}{(2R-r)(4R^2+4Rr+3r^2+r^2-8Rr) - 2Rr^2} =$$

$$\frac{4Rr^2}{(2R-r)\cdot 4(R^2-Rr+r^2) - 2Rr^2} = \frac{2Rr^2}{4R^3+5Rr^2-6R^2r-2r^3}.$$

Thus, remains to prove $\frac{2Rr^2}{4R^3+5Rr^2-6R^2r-2r^3} \geq \frac{r^2}{2R^2-Rr-2r^2} \Leftrightarrow$

$2R(2R^2-Rr-2r^2) \geq 4R^3+5Rr^2-6R^2r-2r^3$.

We have $2R(2R^2-Rr-2r^2) - (4R^3+5Rr^2-6R^2r-2r^3) = r(R-2r)(4R-r) \geq 0$ because $R \geq 2r$ (Eulers Inequality).